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Scaling function for free-electron-laser gain including alternating-gradient focusing

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In the exponential regime before saturation, we present a variational calculation of the gain of a free electron laser (FEL), with alternating-gradient focusing used to supplement the natural focusing of the wiggler. The longitudinal velocity modulation due to the quadrupole focusing is properly taken into account and it is found that it does not prevent quadrupole focusing from efficiently increasing the gain. Our analytic calculation agrees with computer simulation to a few percent, and it provides rapid computation facilitating FEL design optimization.

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There is great interest [1] in the development of free-electron lasers as sources of intense coherent radiation at wavelengths below 1000 Å. For a single pass FEL amplifier, the exponential-folding length of the amplified radiation must be minimized to reduce the required length of the wiggler magnet. This requires an electron beam with high peak current, small normalized emittance, and small energy spread. To achieve high gain at the shortest wavelengths, strong focusing of the electron beam is necessary. In previous work [2] we presented an analysis of the FEL gain including the natural focusing of the wiggler. As emphasized by Scharlemann [3], the natural focusing in a helical wiggler, or in a linear wiggler with parabolic pole faces, does not modulate the electron velocity or phase. On the other hand, alternating-gradient focusing [4–7] can provide stronger focusing and hence potentially higher gain, but it results in a modulation of the longitudinal velocity of each electron, and hence in a modulation of the electron's phase with respect to the electromagnetic wave.

In this Brief Report, we report analytic and numerical calculations of the FEL gain in the exponential regime, including natural and alternating-gradient focusing. We find that despite the longitudinal velocity modulation introduced by the quadrupole focusing, alternating-gradient focusing is effective in achieving increased gain. In fact, we have observed that there is a price paid in the mechanism by which natural focusing results in constant longitudinal velocity. In the case of natural focusing, when an electron moves further from the axis, its transverse angular deviation decreases, while its wiggling amplitude increases in such a manner as to maintain constant longitudinal velocity. As a consequence, for natural focusing, the longitudinal velocities of electrons with large betatron oscillation amplitudes are reduced by a greater amount than in the case of alternating-gradient focusing. The reduction of the dependence of the longitudinal velocity on betatron oscillation amplitude in the case of alternating-gradient focusing turns out to offset the effect of the longitudinal velocity modulation.

Let us comment on the relationship of our work to that of Scharlemann [3]. We consider the exponential gain regime of a short wavelength FEL with a low emittance electron beam from an rf linac. Scharlemann's interest was in big, broad electron beams from induction linacs and to long, tapered wigglers in which most of the output power was provided by trapped electrons. In this case he found that alternating-gradient focusing was very detrimental to the gain, and natural focusing far superior. Our work provides important insight into the exponential regime and shows that in this case alternating-gradient focusing is very effective. This result is not in conflict with Scharlemann's work, but is counter to the intuition one might have had based on his work on a different operating regime.

In earlier work, we presented an analytic calculation [7,8] of the FEL gain valid in the regime of exponential growth before saturation, based upon a dispersion relation derived from the Vlasov-Maxwell equations. This treatment included the effects of the energy spread, emittance, and the natural focusing of the electron beam, and the diffraction and guiding of the radiation field. The dispersion relation was solved using a variational approximation, and the results for the exponential-folding length of the electric field of the fundamental guided mode were expressed in a scaled form. In the present work, we extend our earlier treatment to include alternating-gradient quadrupole focusing. We derive the dispersion relation determining the exponential-folding length and demonstrate an extended form of the scaling relation.

In our calculation, the electron beam's energy distribution $h(\gamma)$ is Gaussian, with average energy $\gamma_0 mc^2$, and rms spread $\gamma_0\sigma$. The static wiggler magnetic field has period λ_w and wave number $k_w = 2\pi/\lambda_w$. The resonant radiation frequency $\omega_r = k_r c$ of the FEL is determined by $k_r = 2\gamma_0^2 k_w/(1+K^2)$, where $K = eB_{\rm rms}/k_w mc$ and $B_{\rm rms}$ is the rms value of the wiggler magnetic field (in mks units). We assume the electron beam focusing is due to both natural focusing of either a helical wiggler or a linear wiggler with parabolic pole faces, and external alternating-gradient quadrupole focusing. We model the external quadrupole focusing by assuming it to result from a magnetic field

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$$\mathbf{B}_Q=(-gy\cos k_Q z, -gx\cos k_Q z, 0),$$

where $k_Q=2\pi/\lambda_Q$ and λ_Q is the period of the external quadrupole field. We assume $\lambda_Q \gg \lambda_w$, and in this case the equations of motion describing the transverse betatron oscillations of the electrons are

$$\frac{d^2x}{dz^2}=-(k_n^2+k_q^2\cos k_Q z)x, \tag{1a}$$

$$\frac{d^2y}{dz^2}=-(k_n^2-k_q^2\cos k_Q z)y, \tag{1b}$$

where $k_n=Kk_w/\gamma\sqrt{2}$ corresponds to natural focusing and $k_q^2=eg/\gamma mc$ to the quadrupole focusing. We assume

$$a\equiv(2k_n/k_Q)^2 \ll 1 \tag{2a}$$

and

$$q\equiv 2(k_q/k_Q)^2 \ll 1 . \tag{2b}$$

In this case, the solutions of the Mathieu equation [9] of Eqs. (1a) and (1b) are well approximated by

$$x \approx x_\beta \cos(k_\beta z+\Phi_x), \tag{3a}$$

$$y \approx y_\beta \cos(k_\beta z+\Phi_y), \tag{3b}$$

where

$$k_\beta=\sqrt{k_n^2+k_q^4/2k_Q^2} . \tag{4}$$

In Eq. (4), k_β is the betatron wave number resulting from both the natural and external quadrupole focusing, and k_n is the wave number of the betatron oscillation, which would exist due to natural focusing in the absence of external quadrupole focusing. The condition for the validity of the approximate solutions of Eqs. (3) and (4) is $\lambda_w \ll \lambda_Q \ll \lambda_q \ll \lambda_\beta \ll \lambda_n$, which can be satisfied in interesting practical cases. The constants $x_\beta, y_\beta, \Phi_x, \Phi_y$ are determined by the initial conditions. Equation (4) explicitly shows the enhancement of focusing due to the quadrupoles. In accelerator physics, this approximation is called the "smooth approximation" [10] and corresponds to neglecting the variation of the betatron functions.

Initially, we assume the electron beam has a uniform longitudinal density, and a uniform "water-bag" distribution inside a four-dimensional sphere in the four-dimensional transverse phase space $\mathbf{R}=(x,y)$, $\mathbf{R}'=d\mathbf{R}/dz=(x',y')$;

$$U(\mathbf{R},\mathbf{R}')=\frac{n_0}{\pi k_\beta^2 R_0^2}\theta(k_\beta^2R_0^2-k_\beta^2R^2-R'^2), \tag{5}$$

where the step function $\theta(v)=1$ for $v>0$ and $\theta(v)=0$ for $v<0$. Integrating $U(\mathbf{R},\mathbf{R}')$ over the angular deviation $\mathbf{R}'$, one obtains the parabolic transverse density profile: $g(R)=n_0(1-R^2/R_0^2)$ for $R<R_0$, and $g(R)=0$ for $R>R_0$. The peak electron density is n_0, and the electron beam current is $I_0=en_0c\pi R_0^2/2$. the rms transverse emittance ϵ of the matched electron beam is defined by

$$\epsilon=(\langle x^2\rangle\langle x'^2\rangle)^{1/2}=(\langle y^2\rangle\langle y'^2\rangle)^{1/2}=k_\beta R_0^2/6 . \tag{6}$$

We consider the linear region before saturation, and write the electric field of the fundamental guided laser mode of frequency ω in the form

$$E(\mathbf{r},\omega)e^{-i\mu k_w z}e^{-i\omega(t-z/c)}\hat{\mathbf{e}}+\text{c.c.} ,$$

where $\hat{\mathbf{e}}$ is either the helical polarization vector $(\hat{\mathbf{x}}+i\hat{\mathbf{y}})/\sqrt{2}$ or the linear polarization vector $\hat{\mathbf{x}}$. The function $E(\mathbf{r},\omega)$ describes the transverse-mode profile in terms of the dimensionless coordinates $\mathbf{r}=\sqrt{2k_rk_w}\mathbf{R}$. The factor $\exp(-i\mu k_w z)$ describes the deviation from free space propagation and $2\pi\mathrm{Im}\mu$ is the growth rate per wiggler period. The exponential-folding length L of the electric field of the amplified mode is given by

$$1/L=k_w\mathrm{Im}\mu . \tag{7}$$

Assuming $\gamma_0 \gg 1$, so that space charge effects are negligible, Vlasov-Maxwell equations have been used to derive a dispersion relation determining μ and $E(\mathbf{r})$:

$$(\nabla_\perp^2+\mu)E(\mathbf{r})=\frac{i}{2}(2\rho\gamma_0)^3\int\frac{d\gamma}{\gamma^2}h'(\gamma)\int d^2p\,u(p^2+k^2r^2)$$
$$\times\int_{-\infty}^0 ds\,e^{-i\alpha s}\exp\left\{-i\frac{b}{2k}\left[\left[r^2+\frac{P^2}{k^2}\right]2ks+\left[r^2-\frac{p^2}{k^2}\right]\sin(2ks)+\frac{\mathbf{p}\cdot\mathbf{r}}{k}2[1-\cos(2ks)]\right]\right\}$$
$$\times E\left[\mathbf{r}\cos ks+\frac{\mathbf{p}}{k}\sin ks\right], \tag{8}$$

where $k=k_\beta/k_w$ characterizes the total focusing strength [Eq. (4)] and the Laplacian $\nabla_\perp^2$ corresponds to the dimensionless coordinates $\mathbf{r}$. In addition, we have defined

$$\alpha=\mu+\frac{\omega-\omega_r}{\omega_r}-2\frac{\gamma-\gamma_0}{\gamma_0}+\frac{p^2+k^2r^2}{4},$$

$$b=-k_q^4/(16k_w^2k_Q^2),$$

$$u(p^2+k^2r^2)=(1/\pi k^2a^2)\theta(k^2a^2-k^2r^2-p^2),$$

$$(2\rho\gamma_0)^3=e^2Z_0n_0K^2[JJ]^2/2mck_w^2,$$

with $[JJ]=1$ for a helical wiggler, and

$$[JJ]=J_0(K^2/2(1+K^2))-J_1(K^2/2(1+K^2))$$

for a linear wiggler. The radius corresponding to the edge of the electron beam expressed in dimensionless coordinates is $a=\sqrt{2k_rk_w}R_0$ and $Z_0=377\ \Omega$ is the impedance of free space.

The dispersion relation of Eq. (8) reduces to that of Eq. (5) of Ref. [2] when the external focusing vanishes, i.e., $b=0$. As in Ref. [2], we note that the dispersion relation of Eq. (8) corresponds to the stationary solutions of the variational form

$$\int d^2r E(\mathbf{r})(\nabla_\perp^2+\mu)E(\mathbf{r})=\int d^2r \int d^2r' E(\mathbf{r})\mathcal{K}(\mathbf{r},\mathbf{r}')E(\mathbf{r}') \tag{9}$$

with the kernal $\mathcal{K}$ now given by

$$\mathcal{K}(\mathbf{r},\mathbf{r}')=(2\rho)^3\int d\gamma h(\gamma)\int_{-\infty}^{0} ds \frac{k^2 s}{\sin^2 ks} u\left[\frac{k^2}{\sin^2 ks}(r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'\cos ks)\right]$$
$$\times e^{-is\{\mu+(\omega-\omega_r)/\omega_r-2[(\gamma-\gamma_0)/\gamma_0]+1/4(k^2/\sin^2 ks)(r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'\cos ks)\}}$$
$$\times e^{-ib/2k\{(2ks/\sin^2 ks)(r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'\cos ks)-(2/\sin ks)[(r^2+r'^2)\cos ks-2\mathbf{r}\cdot\mathbf{r}']\}} .$$

As in Ref. [2], we choose the trial function

$$E(r)=\begin{cases} e^{-\chi r^2/2a^2}, & r\le a \\ AH_0^{(1)}(r\sqrt{\mu}), & r\ge a, \end{cases} \tag{10}$$

where we require $\text{Im}\sqrt{\mu}>0$ to satisfy the boundary condition at $r\to\infty$, and $H_0^{(1)}$ is the Hankel function. The constant A is chosen to assure continuity of $E(\mathbf{r})$. Continuity of the logarithmic derivative at $r=a$ leads to the constraint

$$a\sqrt{\mu}H_0^{(1)\prime}(a\sqrt{\mu})/H_0^{(1)}(a\sqrt{\mu})=-\chi . \tag{11}$$

Employing Eq. (10) in the variational form of Eq. (9) yields

$$Da^2\left[\frac{\mu}{D}\right](1-e^{-\chi})-\chi[1-(1-\chi)e^{-\chi}]$$
$$=\int_{-\infty}^{0}\frac{sds\exp\{-i[\mu/D+(\omega-\omega_r)/\omega_r D]s-(\sigma/D)^2s^2\}}{\cos(ks/D)[1-(i/8\chi)\xi(ka^2)\tan(ks/D)}\left[\frac{1-e^{-\eta_+}}{\eta_+}-\frac{1-e^{-\eta_-}}{\eta_-}\right] \tag{12}$$

with

$$\eta_\pm=\frac{i}{4}(ka^2)\left[\left[\frac{k}{D}\right]s+\frac{\chi}{2}\left[1\mp\cos\left[\frac{ks}{D}\right]\right]\right]$$
$$-\frac{i}{16}\xi(ka^2)\left[\frac{ks}{D}\mp\sin\left[\frac{ks}{D}\right]\right]$$

and

$$\xi=k_q^4/k_Q^2k_\beta^2 ,$$

$$D=\left[\frac{4eZ_0}{\pi mc^2}\frac{K^2}{1+K^2}\frac{I_0}{\gamma_0}\right]^{1/2}[JJ] .$$

The physical significance of the parameter $\xi=k_q^4/k_Q^2k_\beta^2$ can be inferred from Eq. (4), from which we see that

$$\left[\frac{k_n}{k_\beta}\right]^2=1-\frac{\xi}{2} . \tag{13}$$

Thus, ξ determines the fraction of the total focusing strength due to the natural focusing of the wiggler.

Observing that $Da^2=12(k_r\epsilon)(D/k)$, it is seen that the exponential-folding length L of the electric field can be expressed in the scaled form

$$\frac{1}{k_wL}=\text{Im}\mu=DG\left[k_r\epsilon,\frac{\sigma}{D},\frac{k_\beta}{k_wD},\frac{\omega-\omega_r}{\omega_rD},\frac{k_n}{k_\beta}\right] . \tag{14}$$

We have determined the universal gain function $G=\text{Im}(\mu)/D$ [of Eq. (14)] by numerically solving Eqs. (11) and (12). As in our earlier work [2], the scaling law allows the entire physical parameter space to be described by the dependence of G on a few dimensionless scaled variables (five including the effect of external quadrupole focusing). In Fig. 1 we plot $G=\text{Im}(\mu)/D$ against $2k_r\epsilon=4\pi\epsilon/\lambda_r$, for fixed values of scaled energy spread $\sigma/D=0.1$ and scaled total betatron focusing strength $k_\beta/k_wD=0.5$. Curves are shown corresponding to different values of the ratio k_β/k_n, which is determined in terms of ξ by Eq. (14). One sees that in the region of $4\pi\epsilon/\lambda_r\approx 1$, the case of pure natural focusing, $k_\beta/k_n=1$,

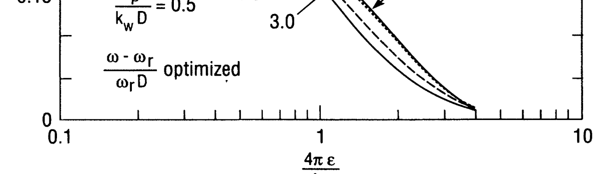

FIG. 1. We plot the scaled growth rate $\text{Im}(\mu)/D$ against $4\pi\epsilon/\lambda$, for fixed values of scaled energy spread $\sigma/D=0.1$ and scaled total betatron focusing strength $k_\beta/k_wD=0.5$. The curves shown correspond to different values of the ratio $k_\beta/k_n=1$, 3, 5, and 10. The detuning is optimized for every point.

has smaller G than the cases corresponding to $k_\beta/k_n > 3$. This demonstrates that alternating-gradient focusing is an efficient method of increasing the gain. As mentioned earlier, an advantage of alternating-gradient focusing is that the reduction of longitudinal velocity with increasing betatron oscillation amplitude is less than for natural focusing. As a consequence, the spread of longitudinal phase drift (relative to the electromagnetic wave) is smaller for alternating-gradient focusing than for natural focusing. This is the reason for G being larger; the smaller the percentage of the total focusing being due to natural focusing. The direct interpretation of Fig. 1 in terms of physical parameters is complicated by the fact that the scaled transverse current D depends on K and γ, and hence varies when the strength of the natural focusing k_n is modified by changing the value of K and/or γ.

In Fig. 1, for each point $(2k_r\epsilon, k_\beta/k_w D, \sigma/D, k_\beta/k_n)$, the scaled detuning $(\omega-\omega_r)/\omega_r D$ was optimized. We note that the optimized detuning criteria (Eq. (13) of Ref. [2]) determined for natural focusing is not valid in the case of alternating-gradient focusing. The optimized detuning is now less than the optimum value for natural focusing, because the longitudinal velocity reduction at the outer edge of the beam is less than in the case of pure natural focusing.

We have compared the results of our variational approximation to those of computer simulation employing a modified version of the code TDA [11]. The modification of TDA consisted of changing the transverse betatron oscillation equations from $x''+k_n^2x=0$, $y''+k_n^2y=0$ to $x''+k_\beta^2x=0, y''+k_\beta^2y=0$, where k_β is given by Eq. (4). The gain lengths determined in these two different ways agree to a few percent. As an example, we consider parameters appropriate to the UV-FEL under design at BNL with $\lambda_r=1000$ Å. In this case

$$\gamma=490\ ,\quad K=1.09\ ,\quad \lambda_w=2.2\ \text{cm}\ .$$

We take the normalized emittance $\epsilon_n=\gamma\epsilon=8$ mm mrad, the energy spread $\sigma=4.35\times10^4$ and the current $I_0=300$ A. The betatron wavelength from natural focusing is $\lambda_n=14$ m. In this case the scaled transverse current $D=1.5\times10^{-2}$. When the focusing is exclusively due to the natural focusing of the wiggler, the power gain length is found to be

$$L/2=1.07\ \text{m (natural focusing)}\ .$$

Using the analytic theory developed in this paper, we have determined the optimum strength of the alternating-gradient focusing to minimize the gain length. The optimum value of the total betatron wavelength $\lambda_\beta=2\pi/k_\beta=3.27$ m. The corresponding optimum value of the power gain length is found to be

$$L/2=0.78\ \text{m (variational)}$$
$$=0.77\ \text{m (modified TDA)}\ .$$

The variational calculation is found to agree with computer simulation to a few percent, and it is seen that the alternating-gradient focusing has led to a 30% reduction in the gain length.

The minimum in the dependence of the gain length on betatron wavelength is quite flat. When $\lambda_\beta=5.88$ m, the power gain length only increases to 0.81 m. In this case the assumed focusing can be realized by choosing the period length of the alternating-gradient structure to be $\lambda_Q=20$ cm and the strength of the quadrupole focusing $k_q=2\pi/\lambda_q$ with $\lambda_q=96$ cm. The inequality constraints necessary for the validity of our approximations are satisfied: $\lambda_w=2.2$ cm $\ll\lambda_Q=20$ cm $\ll\lambda_q=96$ cm $\ll\lambda_\beta=588$ cm $\ll\lambda_n=1400$ cm. The parameter $\xi=1.65$, and the focusing gradient $g=35$ T/m, which is achievable.

Before we developed the analytic theory presented here, properly taking into account the alternating-gradient focusing, we tried to get a rough idea of the effectiveness of external quadrupole focusing using the approximation developed in Ref. [2]. This corresponds to setting $k_n/k_\beta=1$ in Eq. (4), but treating k_n as a free parameter not equal to $Kk_w/\gamma\sqrt{2}$. Doing this ignores the longitudinal velocity modulation and leads to incorrect results. Following this incorrect procedure, we found that the gain length was minimized for $\lambda_\beta=5.7$ m with $L/2=1.00$ m, showing very little improvement over natural focusing. When the longitudinal velocity modulation was properly taken into account, we found the greater improvement stated earlier. The reason for this is that when the longitudinal velocity modulation is taken into account, the dependence of the longitudinal velocity on betatron oscillation amplitude is reduced, leading to smaller spread in the longitudinal phase drift, and hence higher FEL gain.

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